

# Resonance broadening due to particle scattering and mode-coupling in the quasi-linear relaxation of electron beams

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**Abstract.** Of particular interest for radio and hard X-ray diagnostics of accelerated electrons during solar flares is the understanding of the basic non-linear mechanisms regulating the relaxation of electron beams propagating in turbulent plasmas. In this work, it is shown that in addition to scattering of beam electrons, scattering of the beam-generated Langmuir waves via for instance mode-coupling, can also result in broadening of the wave-particle resonance. We obtain a resonance-broadened version of weak-turbulence theory with mode-coupling to ion-sound modes. Resonance broadening is presented here as a unified framework which can quantitatively account for the reduction and possible suppression of the beam instability due to background scattering of the beam electrons themselves or due to scattering of the beam-generated Langmuir waves in fluctuating plasmas. Resonance broadening being essentially equivalent to smoothing of the electron phase-space distribution, it is used to construct an intuitive physical picture for the stability of inverted populations of fast electrons that are commonly observed *in-situ* to propagate in the solar-wind.

## 1. Introduction

Solar flares are transient bursts in the solar atmosphere following the release of energy accumulated in the coronal magnetic field. A major aspect of solar flares is the high acceleration efficiency of electrons inferred from their hard X-ray (HXR) emission [see e.g. *Holman et al.*, 2011; *Kontar et al.*, 2011a, for recent reviews]. Although flare accelerated electrons are prominently diagnosed via their HXR emission at the dense footpoints of coronal magnetic loops, these electrons can also escape into interplanetary space and be detected *in-situ* near-Earth or remotely via their radio emission in the solar wind [e.g. *Lin*, 2011, as a recent review]. As a population of escaping electrons travel further away from the Sun along the interplanetary magnetic field, faster electrons outpace the slower ones creating an electron beam with positive gradient in velocity space. When the number of beam electrons is sufficiently large, the beam becomes unstable to the generation of Langmuir waves which then convert to electromagnetic waves [*Ginzburg and Zhelezniakov*, 1958]. This plasma emission process can be seen via type III radio bursts. The rapid frequency drift of these bursts is associated with the beam travelling away from the Sun in a plasma of decreasing density, see e.g. *Goldman* [1983]; *Melrose* [1985]; *Robinson and Cairns* [1998]; *Melrose* [2009] for reviews. A fundamental problem in the theory of type III solar radio emission is therefore the understanding of the interaction of the exciter with the background plasma which is not quiescent nor homogeneous. For instance, turbulent magnetic fluctuations in the solar wind are known to efficiently scatter non-thermal particles resulting in broad pitch-angle distribution despite the effect of adiabatic focusing of the field, see e.g. *Shalchi* [2009] for a review. These turbulent magnetic

fluctuations also result in cross-field transport of flare accelerated electrons in the corona [Bian *et al.*, 2011; Kontar *et al.*, 2011b] and in the solar wind [Shalchi, 2009]. Beam generated Langmuir waves can also be efficiently scattered by density inhomogeneities. The effect of turbulent density fluctuations has been widely discussed in the context of the beam-plasma relaxation. Vedenov *et al.* [1967] first described refraction by random large-scale density inhomogeneities as a diffusive transfer of Langmuir wave energy in wave-number space. This diffusion equation in wave-number space was then used to describe the spectral transfer of wave-energy out of the region of excitation by the beam [Nishikawa and Ryutov, 1976; Muschietti *et al.*, 1985] providing a path for the weakening of the beam instability and also for acceleration of the electrons during beam-plasma relaxation [Escande, 1979; Kontar *et al.*, 2012; Ratcliffe *et al.*, 2012]. Beam-generated Langmuir waves are shifted toward higher/lower phase velocities when propagating into a region of higher/lower plasma density so that electrons with higher/lower velocities can interact with these waves [Breĭzman and Ryutov, 1969]. Mode-coupling of Langmuir waves, for instance to sound waves, can also play a stabilizing role on the beam plasma system, Papadopoulos *et al.* [1974]; Rowland and Papadopoulos [1977] have suggested that the modulational instability can remove beam-generated Langmuir waves from resonance with the beam by scattering them to different wavenumbers. The important role of density fluctuations is attested by the clumpy spatial distribution of the wave field which is measured *in situ* in the solar wind [Gurnett and Anderson, 1977; Lin *et al.*, 1981] as such localization is probably associated with density cavities [see e.g. Zaslavsky *et al.*, 2010; Hess *et al.*, 2010].

In this work we consider anew this important problem of beam relaxation in turbulent plasmas. We show that resonance broadening provides a unified framework to account for the effects of turbulent scattering of the particles and scattering of the waves in the quasi-linear relaxation of electron beams. Resonance broadening was first discussed by *Dupree* [1966] in the context of quasilinear theory in order to treat certain features associated with the high-intensity of the wave electric field, see also *Rudakov and Tsytovich* [1971]. Independently of its origin, resonance broadening filters out the small scale variations of the electron phase-space distribution and therefore, it provides an intuitive picture for the weakening and possible suppression of the beam instability due to particle and wave scattering.

We start by reviewing in Section 2 major aspects of the quasilinear description of relaxation of electron beams. In Sections 3 and 4, it is shown that the effect of scattering of the particles and waves can be included in the form of broadening of the wave-particle resonance involved in the quasilinear diffusion equations. In Section 5, we start from the *Zakharov* [1972] equations and derive a resonance broadened version of weak-turbulence theory, where broadening of the wave-particle resonance occurs as a result of non-local three-wave coupling between two Langmuir (L) modes and an ion-sound (S) mode with  $k_L, k_{L'} \gg k_S$ . A summary of the results and conclusions are given in Section 6.

## 2. Quasilinear diffusion equations for resonant wave-particle interactions

Quasilinear theory is one of the few theories of plasma turbulence [*Sagdeev and Galeev*, 1969; *Davidson*, 1972; *Melrose*, 1980; *Tsytovich*, 1995]. It is central to the modelling of stochastic acceleration of particles by waves and plasma turbulence, for instance during solar flares [see e.g. *Miller et al.*, 1997; *Petrosian*, 2012; *Bian et al.*, 2012, as reviews].

Originally, quasilinear theory was proposed as a mean-field description of Langmuir wave generation by a population of beam-electrons, also accounting for the self-consistent evolution of the wave spectrum that these electrons may emit and reabsorb in the course of the bump-on-tail instability [*Pines and Schrieffer*, 1962; *Vedenov and Velikhov*, 1963; *Drummond and Pines*, 1964]. The quasilinear diffusion equations describing the relaxation of beam electrons are commonly derived from the Vlasov-Maxwell equations, which for a one dimensional electron plasma read

$$\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{e}{m_e} E(x, t) \frac{\partial}{\partial v} \right] f(x, v, t) = 0, \quad (1)$$

$$\frac{\partial E(x, t)}{\partial t} = -4\pi en \int dv v f(x, v, t). \quad (2)$$

Here,  $f(x, v, t)$  is the electron distribution function (normalized such that  $\int dv f(x, v, t) = 1$ ),  $E(x, t)$  is the electric field, and  $e$ ,  $m_e$  are the electron charge and mass respectively. In this high frequency limit, the ions are forming a static neutralizing background and, hence, the electric current  $j = e \int dv v f(x, v, t)$  is carried entirely by the electrons.

Let us start by considering the problem of the statistical acceleration or deceleration of electrons in a broad spectrum of randomly varying longitudinal electric fields [*Sturrock*, 1966]. The Newton equations of motion for the electrons under the influence of electric fields are

$$\frac{dv}{dt} = \frac{e}{m_e} E(x, t) \ , \quad \frac{dx}{dt} = v \ . \quad (3)$$

The electric field is assumed to be statistically homogeneous and stationary, with the property  $\langle E \rangle = 0$  and  $\langle E^2 \rangle \neq 0$ . For the stochastic acceleration problem at hand, the fluctuating electric field is also characterized by a correlation length  $\lambda$  and a correlation time  $\tau$ . The evolution of the particle distribution function may then be given by a Fokker-

Planck equation,

$$\frac{\partial f(v, t)}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f(v, t)}{\partial v}, \quad (4)$$

where  $D$  is the coefficient involved in the diffusion of electrons in velocity space. Under this diffusion, particles flow down the gradient in velocity space. This tends to flatten this gradient of distribution function and, hence, the action of randomly varying electric fields leads to statistical acceleration of the particles when  $\partial f / \partial v < 0$ , or to their statistical deceleration when  $\partial f / \partial v > 0$ . Accordingly, there is damping or growth of electric energy. The quasilinear approximation is a perturbation scheme based on the smallness of the so-called Kubo number, which is essentially the normalized amplitude of the electric field. Such perturbation scheme is used for obtaining an expression for the diffusion coefficient  $D$  entering Eq.(4) in terms of the spectrum of the electric field fluctuations. The self-consistent rate of damping or growth of the electric field is then determined by applying conservation of total energy, kinetic plus electric in the system, in the particular case of waves having specific dispersion relation  $\omega(k)$  relating the frequency and wave-number involved in the definition of the Fourier components of the electric field fluctuations. This proceeds as follow. The Fourier components  $\hat{E}_{k,\omega}$  of  $E(x, t)$  are defined via

$$E(x, t) = \sum_k \sum_\omega \hat{E}_{k,\omega} e^{i(kx - \omega t)}, \quad (5)$$

with  $k = n\delta k$ ,  $\omega = m\delta\omega$ ,  $\delta k = 2\pi/L$ , and  $\delta\omega = 2\pi/T$ . Here,  $n$  and  $m$  are integers while  $L$  and  $T$  are the spatial and temporal periods of the electric field. Under the *Corrsin* [1959] independence hypothesis, the diffusion coefficient in velocity space can be expressed as [Krommes, 2002]

$$D = \frac{e^2}{m_e^2} \sum_k \sum_\omega |\hat{E}_{k,\omega}|^2 \int_0^\infty dt \langle e^{ikx - i\omega t} \rangle. \quad (6)$$

In the limit of densely packed Fourier harmonic, one is lead to replace the discrete sum by a continuous one  $\sum_{k,\omega} \rightarrow \int dk d\omega / \delta k \delta \omega$  , and since by definition the electric field spectrum is given by

$$S_E(k, \omega) = |\hat{E}_{k,\omega}|^2 / \delta k \delta \omega, \quad (7)$$

one also obtains that for a continuous spectrum of fluctuations,

$$D = \frac{e^2}{m_e^2} \int \int dk d\omega S_E(k, \omega) \int_0^\infty dt \langle e^{ikx - i\omega t} \rangle . \quad (8)$$

In this expression for the diffusion coefficient in velocity space, the function given by

$$R(\omega - kv, \Delta\omega) \equiv \int_0^\infty dt \langle e^{ikx - i\omega t} \rangle, \quad (9)$$

is the resonance function which characterizes the wave-particle interactions. A more precise meaning of the averaging procedure  $\langle \dots \rangle$  and of the resonance width  $\Delta\omega$  involved in this expression for the resonance function will be given below.

We notice that the equations of motion for the particles can also be written in the following non-dimensional form :  $d\tilde{v}/d\tilde{t} = \epsilon \tilde{E}$ ,  $d\tilde{x}/d\tilde{t} = \tilde{v}$ . Time and length have been normalized to  $\tau$  and  $\lambda$  and  $E$  has been normalized to  $\sqrt{\langle E^2 \rangle}$ . The Kubo number  $\epsilon = e\tau^2 \sqrt{\langle E^2 \rangle} / m_e \lambda$  represents the normalized amplitude of the electric field, so that when  $\epsilon \ll 1$ , the resonance function (9) can be calculated via the unperturbed particle trajectory. This unperturbed trajectory is given by  $x = vt$  and  $v = \text{const}$  which are the solutions of the equations of motion for  $\epsilon = 0$ , i.e. in absence of the field. The small Kubo number limit corresponds to the weak-field limit discussed by *Sturrock* [1966]. In absence of background scattering, the  $\epsilon = 0$  zeroth-order particle trajectory is deterministic, hence we also drop the average involved in the resonance function (9). In this resonant case, the integral given by  $\int_0^\infty dt e^{i(kvt - \omega t)}$  is not convergent when  $t \rightarrow \infty$ , so that an infinitesimal



damping factor  $\nu = 0+$  is added in the exponential,

$$\int_0^\infty dt e^{i(kvt-\omega t)} e^{-\nu t} = \frac{i}{(kv-\omega)+i\nu} \xrightarrow{\nu \rightarrow 0} \pi \delta(\omega - kv) + i\mathcal{P}\left(\frac{1}{\omega - kv}\right), \quad (10)$$

where  $\mathcal{P}$  is the principal value. Only the real part contributes to the resonant diffusion coefficient (8) which takes the form

$$D = \frac{\pi e^2}{m_e^2} \int \int dk d\omega S_E(k, \omega) \delta(\omega - kv). \quad (11)$$

A Fourier component  $[\omega, k]$  of the electric field fluctuations can exchange energy only with particles having velocity  $v$  such that the resonance condition  $v = \omega/k$  is fulfilled.

The only linear modes in Equations (1)-(2) are Langmuir waves. Therefore, we now specialize to electric field fluctuations which are produced by Langmuir waves whose dispersion relation is  $\omega(k) \approx \omega_{pe} = \sqrt{4\pi n_e e^2 / m_e}$ , and write  $S_E(k, \omega) = S_E(k) \delta(\omega - \omega_{pe})$ . It is not here important to include the thermal corrections to the dispersion relation for  $k\lambda_{De} \ll 1$ . The spectral energy density of Langmuir waves is given by  $W(k) = 2(S_E(k)/8\pi)$ , where the factor of two accounts for the electric energy and the kinetic energy of the thermal electrons participating in the wave motion. This leads to the following diffusion equation

$$\frac{\partial f(v, t)}{\partial t} = \frac{4\pi^2 e^2}{m_e^2} \frac{\partial}{\partial v} \frac{W(k = \omega_{pe}/v, t)}{v} \frac{\partial f(v, t)}{\partial v}. \quad (12)$$

describing stochastic acceleration of electrons by a spectrum of Langmuir waves. Let us come back to the case of a single Fourier component in the electric field  $\hat{E}_{k,\omega} e^{i(kx-\omega t)} + \text{cc}$ , where cc denotes complex conjugate, so that the diffusion equation is

$$\frac{\partial f(v, t)}{\partial t} = \frac{\pi e^2}{m_e^2} \frac{\partial}{\partial v} |\hat{E}_{k,\omega}|^2 \delta(\omega - kv) \frac{\partial f(v, t)}{\partial v}. \quad (13)$$

For such a single Fourier component, the sharp resonance function  $\delta(\omega - kv)$  involved in

the diffusion equation (13) implies diffusion of the electron distribution function  $f(v)$  over

a single point in velocity space where  $v = \omega/k$ . This is a singular and ill-defined diffusion process. A main consequence of the resonance broadening effect to be discussed below, is that when the resonance function has a finite width  $\Delta\omega$ , even a plane wave can now produce diffusion over a broad region in velocity space. This region in velocity space is centered around  $v = \omega/k$  and has an extent  $\Delta v = \Delta\omega/k$  where  $\Delta\omega$  is the broadening width. However, we know that a plane wave undergoes damping while accelerating the particles, and therefore, the singular diffusion equation (13) can still be used to obtain the average power transferred by the wave to the resonant electrons. The evolution of electron kinetic energy is given by

$$\frac{dE_c}{dt} \equiv \frac{d}{dt} \langle \frac{1}{2} n_e m_e v^2 \rangle = \int dv \frac{\partial f}{\partial t} \frac{1}{2} n_e m_e v^2. \quad (14)$$

Using (13) and one integration by part yields

$$\frac{dE_c}{dt} = - \frac{\pi n_e e^2 |\hat{E}_{k,\omega}|^2 \omega}{m_e k^2} \left. \frac{\partial f}{\partial v} \right|_{v=\omega/k}, \quad (15)$$

consistent with electron acceleration when  $\partial f / \partial v < 0$  or deceleration when  $\partial f / \partial v > 0$ . Moreover, conservation of the energy given by  $d(|\hat{E}_{k,\omega}|^2 / 4\pi + E_c) / dt = 0$ , yields the following equation,  $\partial |\hat{E}_{k,\omega}|^2 / \partial t = 2\gamma |\hat{E}_{k,\omega}|^2$ , describing time evolution of a given Fourier component of the electric field fluctuations. When these electric field fluctuations are produced by Langmuir waves with  $\omega \approx \omega_{pe}$ , the coefficient  $\gamma$  corresponding to Landau damping or growth of the waves is

$$\gamma = \frac{\pi \omega_{pe}^3}{2k^2} \left. \frac{\partial f}{\partial v} \right|_{v=\omega_{pe}/k}. \quad (16)$$

Summing over all Fourier components of the wave field, the evolution of the spectral energy density is also given by

$$\frac{\partial W(k, t)}{\partial t} = 2\gamma W(k, t). \quad (17)$$

Equations (12) and (17), are the standard quasilinear diffusion equations describing the self-consistent evolution of a population of electrons and a spectrum of Langmuir waves these electrons may generate [Pines and Schrieffer, 1962; Vedenov and Velikhov, 1963; Drummond and Pines, 1964]. The coefficient  $\gamma$ , giving the rate of growth or damping of each spectral component of the Langmuir wave electric field, is an average rate of stimulated emission and absorption of the waves by the particles, a point which can be made more transparent in the semi-classical derivation of these quasilinear diffusion equations [Pines and Schrieffer, 1962; Melrose, 1980; Tsytovich, 1995]. The quasilinear diffusion equations generally include terms describing the spontaneous emission of waves and the dynamic friction on the particles.

### 3. Broadening of the wave-particle resonances by particle and wave scattering

The basic rates of wave absorption or emission corresponding to particle acceleration or deceleration are affected by the presence of background scattering of particles or waves in the medium. In the following, the effect of scattering is taken into account in the quasilinear diffusion equations as a broadening of the wave-particle resonance. Indeed, the resonant nature of the interactions can be affected by any physical mechanism which is able to destroy coherence, say on a time scale  $\Delta\omega$ , resulting in broadening of the resonance function, for instance into a Lorentzian form:

$$\pi\delta(\omega - kv) \rightarrow \frac{\Delta\omega}{(\omega - kv)^2 + \Delta\omega^2} . \quad (18)$$

Such Lorentzian form can be obtained by keeping  $\nu$  finite in Equation (10), making it clear the physical interpretation of resonance broadening due to wave damping,

$$\Delta\omega = \nu, \quad (19)$$

i.e., the resonant interaction between an electron and a Langmuir wave is limited by the life-time of the wave. Formally, resonance broadening consists in the substitution

$$\pi\delta(\omega - kv) \rightarrow R(\omega - kv, \Delta\omega), \quad (20)$$

with  $R$  not necessarily a Lorentzian, in the quasilinear diffusion equations. It is here interesting to recall a point made by *Van Vleck and Margenau* [1949] in a related context, which is that line-broadening bridges the gap between resonant and non-resonant absorption processes as the level of scattering in the medium increases. Such distinction between resonant and non-resonant acceleration by waves, involving resonance-broadening as a transition, is also at the core of a recent classification scheme recently proposed by *Bian et al.* [2012] for stochastic acceleration during flares.

In the following section, we discuss two important sources of resonance broadening in the beam-plasma system. These are the scattering of beam electrons and the scattering of beam-generated Langmuir waves in turbulent plasmas.

### 3.1. Resonance broadening due to particle scattering

One main effect of background scattering of the particles on the wave-particle interactions can be understood by modelling such scattering as a additional random perturbation in the original equations of motion :

$$\frac{dv}{dt} = \frac{e}{m_e} E(x, t), \quad \frac{dx}{dt} = v + \zeta(t), \quad (21)$$

where  $\zeta(t)$  is a Gaussian white noise. Scattering enters here as an external source of stochasticity in the equations of motion for the particles. Notice that such scattering does not produce any change in the kinetic energy of the particles, acting also in absence of the electric field. We shall see, however, that scattering affects the particle

acceleration/deceleration rate due to the electric field and, from energy conservation, also modifies the rate of damping/growth of the electric energy. In fact, scattering results in broadening of the wave-particle resonances because it interrupts at random the coherent emission/absorption of the waves by the particles. In this case, one has to deal with the random nature of the integral  $\int_0^\infty dt \langle e^{ikx-i\omega t} \rangle$ , defining the resonance function and, hence, the particle path is written in the form

$$x = vt + \Delta x, \quad (22)$$

where the perturbation  $\Delta x$  accounts for the random scattering of the particles around their free-streaming trajectory. Under the additional assumption that  $\Delta x$  is a Gaussian random process with  $\langle \Delta x^2 \rangle \propto t^\alpha$ , the resonance function becomes broadened around  $\omega - kv$ , in the form of a Lorentzian for  $\alpha = 1$ , of a Gaussian function for  $\alpha = 2$  or in the form of a Airy function for  $\alpha = 3$ .

### 3.1.1. Spatial diffusion

The case  $\alpha = 1$ , with

$$\langle \Delta x^2 \rangle = 2\kappa t, \quad (23)$$

represents a standard spatial diffusion of the particles around the free-streaming trajectory, with a spatial diffusion coefficient  $\kappa$ . For a Gaussian process  $\xi$  with zero average,  $\langle \xi \rangle = 0$ , and finite variance  $\langle \xi^2 \rangle \neq 0$  a cumulant expansion [Kubo, 1962] provides the relation

$$\langle e^\xi \rangle = e^{\langle \xi^2 \rangle / 2}, \quad (24)$$

where the brackets denote averaging over the probability distribution function (PDF) of  $\xi$ , here assumed to be Gaussian. There is a-priori no need to restrict to Gaussian PDFs except as a simplifying assumption. Taking  $\xi = ik\Delta x$ , we obtain that the resonance

function is given by

$$R(\omega - kv, \Delta\omega) = \int_0^\infty dt e^{ikvt - i\omega t} e^{-\Delta\omega t} = \frac{\Delta\omega}{(\omega - kv)^2 + \Delta\omega^2}, \quad (25)$$

where the resonance width  $\Delta\omega$  is given by the diffusive time-scale over one wavelength of the electric field, corresponding to

$$\Delta\omega = \kappa k^2. \quad (26)$$

### 3.1.2. Velocity-space diffusion

Equivalently, the perturbations in the particle path can be written in terms of velocity fluctuations

$$x = (v + \Delta v)t, \quad (27)$$

and, furthermore, we assume that the fluctuations  $\Delta v$  evolve diffusively,

$$\langle \Delta v^2 \rangle = 2Dt, \quad (28)$$

where  $D$  is the diffusion coefficient in velocity space. As a consequence, the resonance function becomes broadened around  $\omega - kv$  as

$$R(\omega - kv, \Delta\omega) = \int_0^\infty dt e^{ikvt - i\omega t} e^{-(\Delta\omega t)^3} = \frac{\pi}{\Delta\omega} C[i(\omega - kv)/\Delta\omega]. \quad (29)$$

This expression involves a Airy function  $C(z)$  which satisfies the equation  $d^2C/dz^2 - zC = \pi^{-1}$  [Gradshteyn and Ryzhik, 1980]. The resonance width is given by

$$\Delta\omega = (k^2 D)^{1/3}. \quad (30)$$

Let us provide few examples of possible sources of velocity-space diffusion. In the original work by Dupree [1966], the source of velocity-space diffusion is the wave electric field itself.

This means that the resonance function (29) involved in the expression for the velocity space diffusion  $D$  given by Eq.(8) is now a function of  $D$ , hence we can rewrite (8) in the form

$$D = \frac{e^2}{m_e^2} \int \int dk d\omega S_E(k, \omega) \int_0^\infty dt e^{ikvt - i\omega t} e^{-k^2 D t^3} . \quad (31)$$

This is a non-linear integral equation for  $D$ . If we take the limit of weak fluctuations amplitude,  $D \rightarrow 0$  in evaluating the resonance function  $R$  then (31) reduces to the familiar quasilinear result (11) with a sharp resonance function, i.e.  $R(\omega - kv, \Delta\omega \rightarrow 0) \rightarrow \pi\delta(\omega - kv)$ . In other words, the quasilinear equations are modified by broadening as the strength of the electric field fluctuations becomes large. Physically, the resonance broadening effect corresponds to secular diffusion of particle trajectories which results from the interaction of the particles with the turbulent electric field fluctuations. We notice that a Gaussian velocity space diffusion with  $\langle \Delta v^2 \rangle \propto t$  leads to Gaussian spatial super-diffusion with  $\langle \Delta x^2 \rangle \propto t^3$ , hence the form of the resonance function (29).

The source of velocity-space diffusion involved in the broadening of the wave-particle resonance may also be unrelated to the wave electric field but instead due to background scattering of the particles for instance at constant kinetic energy. Beam electrons strongly interact with the ambient magnetic field as the latter imposes a direction to their propagation. Moreover, magnetic fluctuations around the mean ambient field produce pitch-angle scattering which acts as an additional source of velocity space diffusion and, hence, of resonance broadening. Pitch-angle scattering results in fluctuations in the partition between translational (along the ambient field) and rotational energy of the particles in a magnetized plasma and when this is due to a spectrum of low-frequency magnetic perturbations, pitch-angle scattering occurs at constant kinetic energy  $E_c$ . A rigorous treatment

of pitch-angle scattering would require a 3D analysis, however let us emphasize few main aspects of it in a 1D model. Let us still call  $v$  the velocity parallel to the magnetic field,  $V = \sqrt{2E_c/m_e}$  the electron speed and  $\mu = v/V$  the pitch-angle cosine. Interactions of the electrons with magnetic field fluctuations produce increments of pitch-angle cosine  $\Delta\mu$  at constant  $V$ , and hence increments of parallel velocity  $\Delta v = V\Delta\mu$ . Squaring the result and averaging leads to

$$\langle \Delta v^2 \rangle = \frac{2E_c}{m_e} \langle \Delta\mu^2 \rangle. \quad (32)$$

Moreover, let us assume that pitch-angle scattering of the particles produces pitch-angle diffusion with

$$\langle \Delta\mu^2 \rangle = 2D^{(\mu)}t, \quad (33)$$

where  $D^{(\mu)}$  is the pitch-angle diffusion coefficient. Hence, pitch angle scattering acts as a source of diffusion in velocity space but at constant kinetic energy  $E_c$ , i.e.

$$\langle \Delta v^2 \rangle = 2 \frac{2E_c D^{(\mu)}}{m_e} t \quad (34)$$

a relation which is analogous to (28). And therefore, the resonance function becomes broadened with a width given by (30) where  $D$  has now the meaning of the coefficient of diffusion for the parallel velocity at constant energy, which according to (34) tantamount to replacing  $D \rightarrow (2E_c/m_e)D^{(\mu)}$  in (30), leading to

$$\Delta\omega = \left( \frac{2E_c D^{(\mu)} k^2}{m_e} \right)^{1/3}, \quad (35)$$

which is the resonance width associated with such scattering mechanism. The above reasoning is due to *Völk* [1973] in his treatment of resonance broadening by pitch-angle scattering. Resonance broadening due to collisions,  $D^{(\mu)}$  taking its collisional value value in (35) is discussed by *Stix* [1992].



### 3.2. Resonance broadening due to wave scattering

While resonance broadening due to scattering of particles has been widely discussed in the past, there is also a possible source of resonance broadening, which is not due to particle scattering but which is instead due to wave scattering.

#### 3.2.1. Wave-number diffusion

For clarity, we first discuss the effect of wave scattering through the following model, which is a variant of the Kubo-Anderson oscillator [*Anderson*, 1954; *Kubo*, 1954]. The model describes the motion of electrons, and their stochastic acceleration or deceleration, in a plane wave electric field whose wave-number is a random function of time:

$$\frac{dv}{dt} = \frac{e}{m_e} \left( \hat{E}_{k,\omega} e^{i[(k+\Delta k)x-\omega t]} + \text{cc} \right), \quad \frac{dx}{dt} = v, \quad \frac{d\Delta k}{dt} = \zeta(t), \quad (36)$$

where, for simplicity,  $\zeta(t)$  is taken to be a Gaussian white noise. According to (8) the expression for the diffusion coefficient in velocity space is given by

$$D = \frac{e^2}{m_e^2} |\hat{E}_{k,\omega}|^2 \int_0^\infty dt e^{ikvt-i\omega t} \langle e^{i\Delta kvt} \rangle. \quad (37)$$

The evolution of the wave-number in this model is a standard diffusion :

$$\langle \Delta k^2 \rangle = 2D^*t, \quad (38)$$

with  $D^*$  the diffusion coefficient in wave-number space. Therefore, we obtain that the diffusion coefficient in velocity space  $D$  is related to the diffusion coefficient in wave-number space  $D^*$  by

$$D = \frac{e^2}{m_e^2} |\hat{E}_{k,\omega}|^2 \int_0^\infty dt e^{ikvt-i\omega t} e^{-D^*v^2t^3}. \quad (39)$$

We have implicitly assumed that the effect of particles scattering is negligibly small compared to that of the waves. Equivalently, the diffusion coefficient in velocity space can be

written in the form

$$D = \frac{e^2}{m_e^2} |\hat{E}_{k,\omega}|^2 R(\omega - kv, \Delta\omega) , \quad (40)$$

with the resonance function still involving the Airy function given by (29) but the resonance width is now given by

$$\Delta\omega = (D^* v^2)^{1/3}, \quad (41)$$

with  $D^*$  the diffusion coefficient in  $k$ -space. While the plane wave is randomly scattered it also undergoes Landau damping when accelerating the particles. The average power transferred to the particles by a wave undergoing random change in  $k$  leads to the coefficient of Landau damping of this wave which can be calculated in a manner similar to what was done for obtaining the Landau damping rate (16) of an unscattered wave. The result is

$$\gamma = \frac{\omega_{pe}^3}{2k^2} \int dv k \frac{\partial f}{\partial v} R(\omega - kv, \Delta\omega), \quad (42)$$

which reduces to the standard expression for Landau damping of a Langmuir wave given by (16) when  $\omega \approx \omega_{pe}$  and when the level of wave scattering becomes small, i.e.  $D^* \rightarrow 0$  or  $\Delta\omega \rightarrow 0$ .

Wave-number diffusion has been widely discussed in the context of the modelling of the propagation of Langmuir waves in a plasma with background density fluctuations. Above, we have shown that such wave-number diffusion leads to broadening of the wave-particle resonances. In the following, we derive a system of resonance-broadened quasilinear diffusion equations which describes the relaxation of an electron beam in a plasma with ambient density fluctuations.

### 3.2.2. Broadening of the wave-particle resonances in a plasma with density fluctuations

The dynamics of both electrons and Langmuir waves are still taken along  $x$ , the direction of the external magnetic field. The plasma density is written as  $n_e[1 + \tilde{n}(x, t)]$  with  $n_e$  the constant background density and  $\tilde{n}(x, t)$  the relative density fluctuation, which is assumed to be weak, i.e.  $\tilde{n}(x, t) \ll 1$ . In general, the characteristic wavenumber  $q$  of low-frequency density fluctuations is much smaller than the characteristic wavenumber  $k$  associated with high-frequency Langmuir waves, so we can make the WKB approximation and treat the Langmuir waves as plasmons or quasi-particles. Ambient density fluctuations are associated with a change in the local refractive index experienced by the waves, so that wave scattering can be modeled by the equations of motion for the Langmuir plasmons, which are given by

$$\frac{dk}{dt} = -\frac{1}{2}\omega_{pe}\frac{\partial\tilde{n}}{\partial x} \equiv F(x, t), \quad (43)$$

$$\frac{dx}{dt} = v_g, \quad (44)$$

where  $F(x, t)$  is the "refraction force" acting on the quasi-particles,  $\omega_{pe} = \sqrt{4\pi n_e e^2/m_e}$  is the local plasma frequency and  $v_g = 3v_{Te}^2 k/\omega_{pe}$  is the group velocity of the quasi-particles. From Equation (43), we see that random refraction induces stochastic change in wavenumber, resulting in a diffusion of the wave energy density in  $k$ -space. An expression for the diffusion coefficient of wave energy in  $k$ -space is thus derived, in the very same quasilinear approximation,

$$D^* = \frac{\omega_{pe}^2 \pi^2}{2} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d\Omega q^2 S_n(q, \Omega) \delta(\Omega - qv_g), \quad (45)$$

where by definition  $\langle \tilde{n}^2 \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d\Omega S_n(q, \Omega)$ . In the particular case where density fluctuations are due to low-frequency compressive waves with a specific dispersion relation  $\Omega = \Omega(q)$ , then  $S_n(q, \Omega) = S_n(q)\delta(\Omega - \Omega(q))$  and  $D^* = (\omega_{pe}^2 \pi^2 / 2) \int_{-\infty}^{\infty} dq q^2 S_n(q) \delta(\Omega(q) - qv_g)$ . A set of quasilinear equations describing the interaction between Langmuir waves and beam electrons in the presence of background density fluctuations is therefore,

$$\frac{\partial f(v, t)}{\partial t} = \frac{4\pi^2 e^2}{m_e^2} \frac{\partial}{\partial v} \frac{W(k = \omega_{pe}/v, t)}{v} \frac{\partial f(v, t)}{\partial v}, \quad (46)$$

$$\frac{\partial W(k, t)}{\partial t} = \frac{\pi \omega_{pe}^3}{k^2} W(k, t) \left. \frac{\partial f(v, t)}{\partial v} \right|_{v=\omega_{pe}/k} + \frac{\partial}{\partial k} D^* \frac{\partial W(k, t)}{\partial k}, \quad (47)$$

which includes a wave-number diffusion in the kinetic equation for the plasmons. As a consequence, this system is equivalent to the following quasilinear diffusion equations with a broadened resonance function :

$$\frac{\partial f(v, t)}{\partial t} = \frac{4\pi e^2}{m_e^2} \frac{\partial}{\partial v} \int dk W(k, t) R(\omega_{pe} - kv, \Delta\omega) \frac{\partial f(v, t)}{\partial v}, \quad (48)$$

$$\frac{\partial W(k, t)}{\partial t} = \frac{\omega_{pe}^3}{k^2} \int dv k \frac{\partial f(v, t)}{\partial v} R(\omega_{pe} - kv, \Delta\omega) W(k, t), \quad (49)$$

where the resonance function is given by (29) and the resonance width is

$$\Delta\omega = (D^* v^2)^{1/3}, \quad (50)$$

with  $D^*$  given by (45).

#### 4. Generalization to three dimensions

Let us consider the generalization of the quasilinear diffusion equations to three dimensions including a broadened resonance function. These are

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \frac{4\pi e^2}{m_e^2} \frac{\partial}{\partial v_i} \int d\mathbf{k} \frac{k_i k_j}{k^2} W_{\mathbf{k}} R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega) \frac{\partial f(\mathbf{v}, t)}{\partial v_j}, \quad (51)$$

describing the diffusion of resonant electrons in velocity space, and

$$\frac{\partial W_{\mathbf{k}}}{\partial t} = \omega_{pe}^3 W_{\mathbf{k}} \int d\mathbf{v} \frac{\mathbf{k}}{k^2} \frac{\partial f(\mathbf{v}, t)}{\partial \mathbf{v}} R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega), \quad (52)$$

giving the rate of growth or damping of spectral energy in the Langmuir waves. They reduce to the standard 3D quasilinear diffusion equations when the broadening width goes to zero, i.e.  $R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega \rightarrow 0) \rightarrow \pi\delta(\omega_{pe} - \mathbf{k} \cdot \mathbf{v})$ . The resonance function appearing in these equations is defined as

$$R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega) = \int dt \langle e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} \rangle. \quad (53)$$

Let us first consider the effect of random perturbations  $\Delta\mathbf{v}$  in the electron velocity, i.e.  $\mathbf{v} \rightarrow \mathbf{v} + \Delta\mathbf{v}$ , hence

$$R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega) = \int dt e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} \langle e^{i\mathbf{k} \cdot \Delta\mathbf{v}t} \rangle = \int dt e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} e^{-\langle (\mathbf{k} \cdot \Delta\mathbf{v})^2 \rangle t^2 / 2}. \quad (54)$$

Introducing the diffusion tensor in velocity space, which is defined via

$$\langle \Delta v_i \Delta v_j \rangle = 2D_{ij}t, \quad (55)$$

we obtain from (54) the following expression for the resonance function

$$R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega) = \int dt e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} e^{-(\Delta\omega t)^2}, \quad (56)$$

where the resonance width  $\Delta\omega$  is given by

$$\Delta\omega = D_{ij}k_i k_j. \quad (57)$$

Summation over repeated indices is implicitly assumed in the previous expression. In the same way, let us consider random perturbations  $\Delta\mathbf{k}$  in the wave-vector, i.e.  $\mathbf{k} \rightarrow \mathbf{k} + \Delta\mathbf{k}$ , hence

$$R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega) = \int dt e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} \langle e^{i\Delta\mathbf{k} \cdot \mathbf{v}t} \rangle = \int dt e^{i\mathbf{k} \cdot \mathbf{v}t - i\omega_{pe}t} e^{-\langle (\Delta\mathbf{k} \cdot \mathbf{v})^2 \rangle t^2 / 2} \quad (58)$$

Introducing the diffusion tensor in wave-vector space,

$$\langle \Delta k_i \Delta k_j \rangle = 2D_{ij}^*t, \quad (59)$$

we can again write the resonance function in the form of a Airy function given by Eq.(56)

but with the resonance width  $\Delta\omega$  given by

$$\Delta\omega = D_{ij}^* v_i v_j. \quad (60)$$

Broadening of the wave-particle resonance occurs here as a result of the random refraction of beam-generated Langmuir waves as they propagate in fluctuating three-dimensional density fluctuations, and therefore,

$$\frac{d\mathbf{k}}{dt} = -\frac{1}{2}\omega_{pe}\nabla\tilde{n}, \quad (61)$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_g, \quad (62)$$

where  $\tilde{n} = n/n_0$  is the relative level of density fluctuations and  $\mathbf{v}_g = 3\lambda_{De}^2\omega_{pe}\mathbf{k}$  is the group velocity. The resonance width (60) involves the diffusion tensor  $D_{ij}^*$  in wave-number space, which is given by

$$D_{ij}^* = \frac{\pi\omega_{pe}^2}{4} \int d\mathbf{q} \int d\Omega q_i q_j S_n(\mathbf{q}, \Omega) \delta(\Omega - \mathbf{q} \cdot \mathbf{v}_g), \quad (63)$$

where by definition  $\langle \tilde{n}^2 \rangle = \int_{-\infty}^{\infty} d\mathbf{q} \int_{-\infty}^{\infty} d\Omega S_n(\mathbf{q}, \Omega)$ , meaning that  $S_n(\mathbf{q}, \Omega)$  is the spectrum of (relative) density fluctuations. We here assume a given level of large-scale density fluctuations of an unspecified origin. The density fluctuations do not have to be produced by waves in which case  $\Omega$  and  $\mathbf{q}$  are not related by a specific dispersion relation  $\Omega(\mathbf{q})$ . However, when the background density fluctuations are produced by low-frequency compressive waves, self-consistency dictates that energy change in the Langmuir quasiparticles is accompanied by energy change in the low-frequency compressive waves resulting in Landau damping/growth of the low-frequency modes on the high-frequency modes, the latter behaving as quasi-particles [Vedenov *et al.*, 1967]. In order to respect energy conservation, the system must be closed by a third equation for the level of compres-

sive density fluctuations which results in  $D_{ij}^*$  becoming a dynamical variable, and, hence, also the resonance width  $\Delta\omega$ , with their time-dependence adjusting self-consistently to the non-linear channeling of the energy between the various modes of motion. In other words, the resonance broadening effect due to Langmuir wave scattering by low-frequency compressive waves is essentially a manifestation of mode coupling. In the following we focus on mode-coupling to ion sound waves but Langmuir wave scattering may be due to other compressive modes, such as kinetic Alfvén waves [*Bian et al.*, 2010; *Bian and Kontar*, 2010]. The spectral properties of these electromagnetic compressive modes are well-documented in the solar wind [*Bale et al.*, 2005; *Mozar and Chen*, 2013].

#### 4.1. Angular scattering only

Before continuing to the topic of resonance broadening by mode-coupling, let us discuss briefly the problem involved, in general, in determining the shape of the resonance function and, in particular, in the case where there is only angular scattering of the particles or only angular scattering of the waves. The resonance function entering the quasilinear diffusion equations involves the quantity  $\omega_{pe} - \mathbf{k} \cdot \mathbf{v} = \omega_{pe} - kv[\sin\theta \sin\theta'(\cos\phi \cos\phi' + \sin\phi \sin\phi') + \cos\theta \cos\theta']$ , which is written here in the spherical coordinate systems  $(v, \theta, \phi)$  and  $(k, \theta', \phi')$  in velocity space and wave-number space, respectively. An essential problem arises when trying to determine the shape of the resonance function when diffusion occurs only in angle,  $\theta$  for the particles or  $\theta'$  for the waves. The problem is that none of these diffusion processes are characterized by Gaussian PDFs. This is quite obvious by noticing that these PDFs must be periodic functions of the angles involved and, therefore, cannot be Gaussian. There are four angles involved in the problem and they can be reduced to two by restricting our considerations to not too large angular spread of the particles or the

waves around  $\theta \sim 0$  or  $\theta' \sim 0$  corresponding to the direction of the mean field, i.e.

$$\omega_{pe} - \mathbf{k} \cdot \mathbf{v} = \omega_{pe} - kv \cos \theta \cos \theta' = \omega_{pe} - kv \left(1 - \frac{\theta^2}{2}\right) \left(1 - \frac{\theta'^2}{2}\right). \quad (64)$$

Let us first consider angular scattering of the particles assuming a narrow distribution of waves having  $\theta' = 0$ , such that

$$\omega_{pe} - \mathbf{k} \cdot \mathbf{v} = \omega_{pe} - kv \left(1 - \frac{\theta^2}{2}\right). \quad (65)$$

Pitch-angle diffusion of the particles results in that  $P(\theta, t)$ , the PDF for the pitch-angle  $\theta$  evolves according to the diffusion equation

$$\frac{\partial P(\theta, t)}{\partial t} = \frac{1}{\theta} \frac{\partial}{\partial \theta} \theta D_{\theta\theta} \frac{\partial P(\theta, t)}{\partial \theta}, \quad (66)$$

when  $\sin \theta \sim \theta$ . Here,  $D_{\theta\theta}$  is the component of the diffusion tensor corresponding to particle scattering in  $\theta$  only. The solution of this equation can be approximated by a Gaussian only for  $\theta \sim 0$  and when  $D_{\theta\theta}t \ll 1$ , in which case

$$\langle \theta^2 \rangle = 2D_{\theta\theta}t. \quad (67)$$

This the domain of validity of the approximate approach discussed in Section III above, for the treatment in 1D of resonance broadening by pitch-angle scattering of the particles, where we had introduced instead of  $\theta$  the notation  $\mu = \cos \theta$  for the pitch-angle cosine of the particles. Outside of this domain of validity, analytical determination of the resonance function must rely on the more accurate PDF  $P(\mu, t)$  which is known to be given by series expansion in terms of Legendre polynomials. The averaging over  $P(\mu, t)$  which is involved in the determination of the resonance function becomes a more complicated analytical procedure, probably amenable to numerical solution only. However, the above approximate treatment provided us with an estimate for the resonance width, despite that

the precise "line-shape" may not well be the correct one.



The perfectly symmetric situation of angular scattering of the waves at constant modulus  $k$  requires similar considerations, as in this case

$$\omega_{pe} - \mathbf{k} \cdot \mathbf{v} = \omega_{pe} - kv(1 - \frac{\theta'^2}{2}), \quad (68)$$

and angular diffusion of the waves results in that  $P(\theta', t)$ , the PDF for the angle  $\theta'$  evolves according to the diffusion relation

$$\langle \theta'^2 \rangle = 2D_{\theta'\theta'}^* t, \quad (69)$$

where  $D_{\theta'\theta'}^*$  is the component of the diffusion tensor corresponding to wave scattering in  $\theta'$  only. Treatment of the simultaneous scattering of waves and particles is also not quite a simple task as this requires determination of the joint PDF  $P(\theta, \theta', t)$  in order to carry the averaging procedure with respects to these random variables.

## 5. Resonance broadening due to mode-coupling of Langmuir waves with ion sound waves and weak-turbulence theory

Let us focus on the specific situation of coupling to the ion-sound waves described by the *Zakharov* [1972] equations

$$\frac{i}{\omega_{pe}} \frac{\partial E}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla^2 E = \left( \frac{n}{2n_0} \right) E, \quad (70)$$

$$n_0 m_i \left[ \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right] \left( \frac{n}{n_0} \right) = \frac{1}{4\pi} \nabla^2 |E|^2, \quad (71)$$

where  $c_s$  is the sound speed and  $\lambda_{De}$  is the Debye length. The first equation is a Schrodinger equation describing the evolution of the Langmuir wave electric field in low-frequency density inhomogeneities. The second equation describes the dynamical evolution of these density inhomogeneities when the latter are assumed to be produced by sound waves. These wave equations are coupled through the terms on their right-hand

side. These terms describe refraction and ponderomotive effects, e.g. [Thornhill and Ter Haar, 1978; Robinson, 1997]. The Zakharov equations exploit the separation of time scales between the dynamics of Langmuir and ion-sound waves, so that the ponderomotive force represents in fact the time average back-reaction of the high-frequency oscillations on the low-frequency density inhomogeneities. The ponderomotive force is the spatial gradient of a potential, i.e.  $\mathbf{f} = \nabla U$  with  $U = |E|^2/8\pi n_0$ , and this force can be shown to act primarily on the electrons. In addition to the separation of time-scales inherent to the Zakharov equations, one can further exploit a separation of length-scales by considering the particular case of non-local three-wave interactions involving an ion-sound mode having a wave-vector and frequency given by  $[\mathbf{q}, \Omega(\mathbf{q})]$  and two Langmuir modes,  $[\mathbf{k}, \omega(\mathbf{k})]$  and  $[\mathbf{k} + \delta\mathbf{k}, \omega(\mathbf{k} + \delta\mathbf{k})]$ , under the restriction  $\delta k \ll k$  for non-local mode coupling. Therefore, the two resonance conditions for mode coupling are  $\mathbf{q} + \mathbf{k} = \mathbf{k} + \delta\mathbf{k}$  and  $\Omega(\mathbf{q}) + \omega(\mathbf{k}) = \omega(\mathbf{k} + \delta\mathbf{k})$ . By Taylor expanding the second condition, we obtain that  $\Omega(\mathbf{q}) = \delta\mathbf{k} \cdot \nabla_{\mathbf{k}} \omega(\mathbf{k})$  and using the first one, which is  $\mathbf{q} = \delta\mathbf{k}$ , the resonance conditions for the non-local three-wave interactions are found to reduce to only one condition  $\Omega(\mathbf{q}) = \mathbf{q} \cdot \mathbf{v}_g$ , i.e.  $\pm qc_s = 3\lambda_{De}^2 \omega_{pe} \mathbf{q} \cdot \mathbf{k}$ , which is similar to a wave-particle resonance condition  $\omega(\mathbf{k}) = \mathbf{k} \cdot \mathbf{v}$ . Non-local resonant interactions between the large-scale and the small-scale waves (quasi-particles) are therefore expected to mediate local diffusive energy transfer among the small-scale waves in analogy with wave-particle interactions. By local diffusive energy transfer we mean that the spread of spectral energy associated with the small-scale waves involves small steps in wave-number space, having  $\delta k \ll k$  and that this spread is a diffusion. Since the first Zakharov equation has the form of a Schrodinger equation describing the dynamics of Langmuir waves in the a slowly varying potential  $n/n_0$ ,

it is standard to apply WKB analysis to it. In order to do that it is convenient to use a generalization of the Fourier transform to spatially inhomogeneous systems [Gershgorin *et al.*, 2009]. The window transform of  $E(\mathbf{x}, t)$  is defined by

$$\Gamma[E(\mathbf{x})] \equiv \hat{E}(\mathbf{x}, \mathbf{k}) = \int d\mathbf{x}_0 w(\epsilon^*|\mathbf{x} - \mathbf{x}_0|) E(\mathbf{x}_0) e^{i\mathbf{k} \cdot \mathbf{x}_0}, \quad (72)$$

which is a kind of wavelet transform that in the case where the window function  $w(x) = \exp(-x^2)$  is called a Gabor transform. The parameter  $1/\epsilon^*$  is the width of the window which is taken to be much smaller than the characteristic length of the inhomogeneity but much larger than the wavelength of the Langmuir waves that propagate in the inhomogeneous medium. Note that the Gabor transform can be viewed as a localized Fourier transform around  $x$  with support  $1/\epsilon^*$ : when  $\epsilon^*$  goes to zero, the filtering width goes to infinity and the Gabor transform becomes a Fourier transform. Applying this transform to Equation (70), we have

$$\frac{i}{\omega_{pe}} \frac{\partial \hat{E}_{\mathbf{k}, \mathbf{x}}}{\partial t} = \left[ \frac{3}{2} \lambda_{De}^2 k^2 + \left( \frac{n}{2n_0} \right) - \nabla \left( \frac{n}{2n_0} \right) \right] \hat{E}_{\mathbf{k}, \mathbf{x}} + i \nabla \left( \frac{n}{2n_0} \right) \cdot \frac{\partial \hat{E}_{\mathbf{k}, \mathbf{x}}}{\partial \mathbf{k}} - 3 \lambda_{De}^2 i \mathbf{k} \cdot \nabla \hat{E}_{\mathbf{k}, \mathbf{x}} \quad (73)$$

Multiplying by  $\hat{E}_{\mathbf{k}, \mathbf{x}}^*$ , the imaginary part of (73) gives the Liouville equation for the small-scale wave energy density. Information on the phases of the small-scale waves is now lost as these waves are treated as quasi-particles. Moreover, let us introduce the wave-action density, defined as

$$N_{\mathbf{k}} = \frac{W_{\mathbf{k}, \mathbf{x}}}{\omega(\mathbf{k})}, \quad (74)$$

and the group velocity  $\mathbf{v}_g = 3 \lambda_{De}^2 \omega_{pe} \mathbf{k}$ , therefore, the WKB limit of the Zakharov equations are written as in Vedenov *et al.* [1967]:

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \mathbf{v}_g \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{x}} - \frac{1}{2} \omega_{pe} \nabla \left( \frac{n}{n_0} \right) \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} = 0, \quad (75)$$

$$n_0 m_i \left[ \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right] \left( \frac{n}{n_0} \right) = \frac{1}{2\omega_{pe}} \nabla^2 \int d\mathbf{k} N_{\mathbf{k}}. \quad (76)$$

The *Zakharov* [1972] equation were first written by *Vedenov et al.* [1967] in this WKB form, which corresponds to restricting the consideration to non-local three-wave coupling with  $k_L, k_{L'} \sim k \gg k_S = q$  in the process  $L + S = L'$ . This system is similar to the field/particle system of Eqs.(1)-(2) we started from, so that quasilinear diffusion equations can be derived from it. Linearizing the first equation (75) with respect to  $N_{\mathbf{k}} = N_{\mathbf{k}}^0 + \delta N_{\mathbf{k}}$ ,  $i(\Omega(\mathbf{q}) - \mathbf{q} \cdot \mathbf{v}_g) \delta N_{\mathbf{k}} = (\omega_{pe} n / 2n_0) \mathbf{q} \cdot \partial N_{\mathbf{k}}^0 / \partial \mathbf{k}$ , and substituting  $\delta N_{\mathbf{k}}$  in the second equation, one obtains the dispersion relation connecting the frequency  $\Omega(\mathbf{q})$  and the wave-vector  $\mathbf{q}$  of the ion-acoustic wave which now includes an imaginary part  $\gamma_{\mathbf{q}}$ . By considering the imaginary part  $\gamma_{\mathbf{q}}$  of  $\Omega(\mathbf{q})$  to be small compared with the frequency  $\Omega(\mathbf{q})$ , i.e.  $\gamma_{\mathbf{q}} \ll \Omega(\mathbf{q})$  then

$$\gamma_{\mathbf{q}} = \frac{\pi \omega_{pe}^2 q}{8n_0 m_i c_s} \int d\mathbf{k} \mathbf{q} \cdot \frac{\partial N_{\mathbf{k}}^0}{\partial \mathbf{k}} \delta(\Omega(\mathbf{q}) - \mathbf{q} \cdot \mathbf{v}_g), \quad (77)$$

where in the last expression  $\Omega(\mathbf{q}) = \pm q c_{cs}$  is used as dispersion relation. The imaginary part  $\gamma_{\mathbf{q}}$  is the coefficient describing the Landau damping or growth of the ion-acoustic waves on the Langmuir plasmons. When it is positive the coefficient (77) is the growth rate of the so-called modulational instability [*Thornhill and Ter Haar*, 1978].

Moreover, we have seen above that Langmuir plasmons undergo a diffusion in wavenumber space in the presence of a spectrum of density fluctuations. Here, this diffusion process in wave-number space occurs as a result of the action of the stochastic refraction force of the sound modes on the quasi-particles. The diffusion coefficient is given by  $D_{ij}^* = (\pi \omega_{pe}^2 / 4) \int d\mathbf{q} q_i q_j S_n(\mathbf{q}) \delta(\Omega(\mathbf{q}) - \mathbf{q} \cdot \mathbf{v}_g)$ . Therefore, in the non-local limit and under

the random phase approximation, the Zakharov equations take the common quasilinear diffusive form given by

$$\frac{\partial N(\mathbf{k})}{\partial t} = \frac{\partial}{\partial k_i} D_{ij}^* \frac{\partial N(\mathbf{k})}{\partial k_j}, \quad (78)$$

$$\frac{\partial S_n(\mathbf{q})}{\partial t} = \gamma_{\mathbf{q}} S_n(\mathbf{q}). \quad (79)$$

On the other hand, we have seen that wave-number diffusion is a source of broadening of the wave-particle resonance. Therefore, the quasilinear diffusion equations describing the interactions between Langmuir waves and electrons will involve a resonance function  $R(\omega - \mathbf{k} \cdot \mathbf{v}, \Delta\omega)$  which is broadened due to the mode-coupling of Langmuir waves with sound waves described by Eqs.(78)-(79). As shown in Section 4, the wave-particle resonance width is given by  $\Delta\omega = (D_{ij}^* v_i v_j)^{1/3}$ .

To summarize, we have obtained the following resonance-broadened model of weak-turbulence:

$$\frac{\partial f(\mathbf{v})}{\partial t} = \frac{\partial}{\partial v_i} D_{ij} \frac{\partial f(\mathbf{v})}{\partial v_j}, \quad (80)$$

$$\frac{\partial N(\mathbf{k})}{\partial t} = \gamma_{\mathbf{k}} N(\mathbf{k}), \quad (81)$$

$$\frac{\partial S_n(\mathbf{q})}{\partial t} = \gamma_{\mathbf{q}} S_n(\mathbf{q}), \quad (82)$$

where

$$D_{ij} = \frac{4\pi e^2 \omega_{pe}}{m_e^2} \int d\mathbf{k} \frac{k_i k_j}{k^2} N(\mathbf{k}) R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega), \quad (83)$$

$$\gamma_{\mathbf{k}} = \omega_{pe}^3 \int d\mathbf{v} \frac{\mathbf{k}}{k^2} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega), \quad (84)$$

$$\gamma_{\mathbf{q}} = \frac{\pi \omega_{pe}^2 q}{8n_0 m_i c_s} \int d\mathbf{k} \mathbf{q} \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} \delta(\Omega(\mathbf{q}) - \mathbf{q} \cdot \mathbf{v}_g). \quad (85)$$

and where the wave-particle resonance width entering  $R(\omega_{pe} - \mathbf{k} \cdot \mathbf{v}, \Delta\omega)$  is given by

$$\Delta\omega = (D_{ij}^* v_i v_j)^{1/3}, \quad (86)$$

which is related to the dynamical level of sound waves fluctuations through

$$D_{ij}^* = \frac{\pi\omega_{pe}^2}{4} \int d\mathbf{q} q_i q_j S_n(\mathbf{q}) \delta(\Omega(\mathbf{q}) - \mathbf{q} \cdot \mathbf{v}_g), \quad (87)$$

where  $\Omega(\mathbf{q}) = \pm qc_s$ . In this formulation of the *Vedenov et al.* [1967] equations, the effect of mode-coupling between the Langmuir plasmons and the sound-waves is taken into account through a broadening of the wave-particle resonances.

Notice that the level of sound waves  $S_n(\mathbf{q})$  entering (87), which is also involved in the expression for the broadening width (86), is depending on time via Eq.(82) which involves  $\gamma_{\mathbf{q}}$ , and hence  $\partial N_{\mathbf{k}}/\partial \mathbf{k}$ . This dynamical broadening of the wave-particle interaction reflects the self-consistent level of Langmuir wave scattering by an evolving spectrum of sound modes in the beam-plasma system. Disregarding time evolution of the sound waves given by Eq.(82) results in Eqs.(80)-(81), together with Eqs.(83)-(84) and with Eqs.(86)-(87), being a description of the relaxation of an electron beam in a *prescribed* level of density fluctuations unaffected by the Langmuir waves. These density fluctuations may not necessarily be produced by sound waves for an arbitrary dispersion relation  $\Omega(\mathbf{q})$ . These density fluctuations may not necessarily be produced by waves when no such dispersion relation exists, in which case we formally need to take  $D_{ij}^* = \pi\omega_{pe}^2/4 \int d\mathbf{q} \int d\Omega q_i q_j S_n(\mathbf{q}, \Omega) \delta(\Omega - \mathbf{q} \cdot \mathbf{v}_g)$  instead of (87) in the expression for the wave-particle resonance width (86).

## 6. Summary and discussion

In general, the resonance function of quasilinear theory has a finite width and can be written in the form given by

$$R(\omega - kv, \Delta\omega) = \int_0^\infty dt \langle e^{ikx - i\omega t} \rangle, \quad (88)$$

where  $\Delta\omega$  is the resonance width. The quasilinear diffusion coefficient then takes the form

$$D = \frac{e^2}{m_e^2} \int \int dk d\omega S_E(k, \omega) R(\omega - kv, \Delta\omega), \quad (89)$$

where  $S_E(k, \omega)$  is the spectrum of electric field fluctuations. We are interested in electric field fluctuations which are produced by Langmuir waves whose dispersion relation is  $\omega(k) \approx \omega_{pe} = \sqrt{4\pi n_e e^2 / m_e}$ , and hence we can write  $S_E(k, \omega) = S_E(k) \delta(\omega - \omega_{pe})$ . The spectral energy density of Langmuir waves is given by  $W(k) = 2(S_E(k)/8\pi)$ , where the factor of two accounts for the electric energy and the kinetic energy of the thermal electrons participating in the wave motion. Hence, the quasilinear diffusion coefficient is given by

$$D = \frac{4\pi e^2}{m_e^2} \int dk W(k) R(\omega_{pe} - kv, \Delta\omega). \quad (90)$$

By conservation laws, statistical acceleration of the particles by the waves, or their deceleration, means damping or growth of the waves. Therefore, the rate of Landau damping or growth of Langmuir waves is given by

$$\gamma = \frac{\omega_{pe}^3}{2k^2} \int dv k \frac{\partial f}{\partial v} R(\omega_{pe} - kv, \Delta\omega). \quad (91)$$

The quasilinear diffusion equations, involving a broad resonance function, have the standard form given by the system

$$\frac{\partial f(v)}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f(v)}{\partial v}, \quad (92)$$

$$\frac{\partial W(k)}{\partial t} = 2\gamma W(k), \quad (93)$$

which by construction respects conservation of the energy. In practice, it is useful to approximate the broad resonance function, entering the quasilinear diffusion equations, by a square function

$$R(\omega_{pe} - kv, \Delta\omega) = \begin{cases} \frac{\pi}{\Delta\omega}, & |\omega_{pe} - kv| < \Delta\omega, \\ 0, & |\omega_{pe} - kv| > \Delta\omega, \end{cases} \quad (94)$$

having the property that  $R(\omega_{pe} - kv, \Delta\omega) \xrightarrow{\Delta\omega \rightarrow 0} \pi\delta(\omega_{pe} - kv)$ , the sharp resonance being obtained from  $x = vt$  and  $\omega = \omega_{pe}$  in (88) .

Turbulent scattering of particles results in velocity fluctuations  $\Delta v$  around the free-streaming trajectory so that we have instead  $x = (v + \Delta v)t$  in (88). When these fluctuations evolve diffusively,  $\langle \Delta v^2 \rangle = 2Dt$ , then the width of the resonance is given by

$$\Delta\omega = (Dk^2)^{1/3}. \quad (95)$$

In *Dupree* [1966] work  $D$  is the quasilinear diffusion coefficient itself, so that  $\Delta\omega$  is a function of the electric field amplitude. Pitch-angle scattering off turbulent magnetic field fluctuations can also broaden the wave-particle resonance leading to  $\Delta\omega = (2E_c D^{(\mu)} k^2 / m_e)^{1/3}$  which tantamount to replacing  $D \rightarrow (2E_c / m_e) D^{(\mu)}$  in (95), with  $D^{(\mu)}$  the pitch-angle diffusion coefficient. Resonance broadening due to collisional pitch-angle scattering,  $D^{(\mu)}$  taking its collisional value, is discussed by *Stix* [1992]. Langmuir wave scattering off background density fluctuations can also result in broadening of the resonance. In particular, when the beam-driven Langmuir wave undergoes stochastic refraction, its wave-number evolves diffusively so that  $k \rightarrow k + \Delta k$  in (88). Assuming diffusive evolution in wave-number space  $\langle \Delta k^2 \rangle = 2D^*t$ , the resonance width is then given by

$$\Delta\omega = (D^* v^2)^{1/3}, \quad (96)$$

where  $D^*$  is the diffusion coefficient in wave-number space. Finally, let us discuss the consequence of resonance broadening on the development of the beam instability. From the broadening frequency  $\Delta\omega$ , one can also define a scale  $\Delta v$  in velocity-space given by

$$\Delta v = \frac{\Delta\omega}{k}. \quad (97)$$



Using the square as a representation of the resonance function, the growth rate of the beam-plasma instability can also be written in terms of  $\Delta v$ , as

$$\gamma(\Delta v) \approx \frac{\pi\omega_{pe}^3}{2k^2} \frac{f(\omega_{pe}/k + \Delta v/2) - f(\omega_{pe}/k - \Delta v/2)}{\Delta v} \equiv \frac{\pi\omega_{pe}^3}{2k^2} \frac{\Delta f}{\Delta v}. \quad (98)$$

In other words, the basic time-scale  $\gamma$  for the development of the beam instability is now a function of the resonance width  $\Delta\omega$  through  $\Delta v$ . Resonance broadening filters out variations in the electron phase-space distribution which are smaller than  $\Delta v$ . As phase-space derivatives also yield the rate of growth of the beam instability in Eq.(98),  $\gamma$  will in turn depend on which scale the phase-space has been filtered by the underlying broadening processes. Notice that when  $\Delta v \rightarrow 0$ , the expression for the growth rate Eq.(98) including broadening reduces to the standard Landau damping/growth rate, i.e.  $\gamma(\Delta v \rightarrow 0) \rightarrow (\pi\omega_{pe}^3/2k^2)\partial f/\partial v|_{v=\omega_{pe}/k}$ . This means that for a given distribution function  $f(v)$  including a beam of extent  $\Delta v_b$ , the growth rate  $\gamma(\Delta v)$  can either be positive or negative depending on the value of the filtering scale  $\Delta v$  in velocity space as illustrated below. Let us consider qualitatively the situation of the beam instability depicted in Fig 1. Due to the presence of the electron beam, Langmuir waves are expected to grow with a phase speed of the order of the beam speed  $\sim v_b$ . However, the presence of background scattering has the effect on broadening the resonant region in velocity space. For small resonance width  $\Delta v = v_1 - v_2 \ll \Delta v_b$ , the average slope of electron phase-space distribution is still positive, therefore the particles loose energy and the waves grow but at a reduced rate. The effective growth rate is given by Equation (98) for  $\gamma(\Delta v)$ . For a large resonance width i.e. when  $\Delta v = v_3 - v_4 \gg \Delta v_b$ , the average slope becomes positive and the waves damp as now  $\gamma(\Delta v) < 0$ . In the situation of Fig (1), the condition  $\Delta v \gg \Delta v_b$  sets the regime of the suppression of the beam instability due to the presence of

background scattering of the particles or of the waves. The growth rate (98) turns out to be negative despite the existence of a region of positive slope for the distribution function, because small-scale features in velocity space, smaller than  $\Delta v$ , and which should act as a reservoir of free-energy for the Langmuir waves to grow, have been effectively filtered out. Resonance broadening is a general mechanism at the origin of the weakening and possible suppression of the beam-plasma instability. As a consequence, it can be used as a tool to explain stable inverted populations of fast electrons commonly observed in-situ in the solar-wind (P. Kellogg, private communication). We refer to the work by *Lin et al.* [1986], where electron distribution functions are observed not to exhibit the strong plateauing predicted by quasi-linear models.

With respect to Langmuir waves scattering off background density fluctuations, we obtain the following criterium for the suppression of the beam instability :

$$D^* > \frac{\omega_{pe}^3}{v_b^2} \left( \frac{\Delta v_b}{v_b} \right)^3 \quad (99)$$

When needed, a more accurate criterium can be obtained by directly solving the marginal stability  $\gamma(\Delta\omega) = 0$  condition, Equation (98).

To summarize, we have shown that resonance broadening provide a unified framework accounting for both the effects of particle and wave scattering during the quasilinear relaxation of electron beams in turbulent plasmas. Resonance broadening is essentially equivalent to filtering-out small scale features of the electron phase-space distribution. As a beam is such a small scale feature in velocity space, resonance broadening provides an intuitive picture for the stabilization of the beam instability. Starting directly from the electrostatic Zakharov equations, we derived a resonance broadened version of weak-turbulence theory, where broadening of the wave-particle resonance occurs as a result of

non-local mode coupling to ion-sound waves. In this case, broadening is non-linear and dynamical reflecting the self-consistent level of ion-sound waves generated by Langmuir waves in the system. In the future, we plan to extend this formalism to account for the mode-coupling of beam generated Langmuir waves to kinetic Alfvén waves.

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## References

- Anderson, P. W. (1954), A Mathematical Model for the Narrowing of Spectral Lines by Exchange or Motion, *Journal of the Physical Society of Japan*, *9*, 316.
- Bale, S. D., P. J. Kellogg, F. S. Mozer, T. S. Horbury, and H. Reme (2005), Measurement of the Electric Fluctuation Spectrum of Magnetohydrodynamic Turbulence, *Physical Review Letters*, *94*(21), 215002, doi:10.1103/PhysRevLett.94.215002.
- Bian, N., A. G. Emslie, and E. P. Kontar (2012), A Classification Scheme for Turbulent Acceleration Processes in Solar Flares, *Astrophys. J.*, *754*, 103, doi:10.1088/0004-637X/754/2/103.
- Bian, N. H., and E. P. Kontar (2010), A gyrofluid description of Alfvénic turbulence and its parallel electric field, *Physics of Plasmas*, *17*(6), 062,308–+, doi:10.1063/1.3439682.
- Bian, N. H., E. P. Kontar, and J. C. Brown (2010), Parallel electric field generation by Alfvén wave turbulence, *Astron. Astrophys.*, *519*, A114+, doi:10.1051/0004-6361/201014048.

- Bian, N. H., E. P. Kontar, and A. L. MacKinnon (2011), Turbulent cross-field transport of non-thermal electrons in coronal loops: theory and observations, *Astron. Astrophys.*, *535*, A18, doi:10.1051/0004-6361/201117574.
- Breĩzman, B. N., and D. D. Ryutov (1969), Quasilinear Relaxation of an Electron Beam in an Inhomogeneous Bounded Plasma, *Soviet Journal of Experimental and Theoretical Physics*, *30*, 759–+.
- Corrsin, S. (1959), Lagrangian Correlation and some Difficulties in Turbulent Diffusion Experiments, *Advances in Geophysics*, *6*, 441–+.
- Davidson, R. C. (1972), *Methods in nonlinear plasma theory.*, 356 p. pp., New York, NY (USA): Academic Press.
- Drummond, W. E., and D. Pines (1964), Nonlinear plasma oscillations, *Annals of Physics*, *28*, 478–499, doi:10.1016/0003-4916(64)90205-2.
- Dupree, T. H. (1966), A Perturbation Theory for Strong Plasma Turbulence, *Physics of Fluids*, *9*, 1773–1782, doi:10.1063/1.1761932.
- Escande, D. F. (1979), One-dimensional kinetic beam-plasma instability in a fluctuating plasma, *Physics of Fluids*, *22*, 321–331, doi:10.1063/1.862583.
- Gershgorin, B., Y. V. Lvov, and S. Nazarenko (2009), Canonical Hamiltonians for waves in inhomogeneous media, *Journal of Mathematical Physics*, *50*(1), 013,527, doi:10.1063/1.3054275.
- Ginzburg, V. L., and V. V. Zhelezniakov (1958), On the Possible Mechanisms of Sporadic Solar Radio Emission (Radiation in an Isotropic Plasma), *Sov. Astronom. Zh.*, *35*, 694.
- Goldman, M. V. (1983), Progress and problems in the theory of type III solar radio

- emission, *Solar Phys.* , *89*, 403–442, doi:10.1007/BF00217259.
- Gradshteyn, I. S., and I. M. Ryzhik (1980), *Table of integrals, series and products*, New York: Academic Press.
- Gurnett, D. A., and R. R. Anderson (1977), Plasma wave electric fields in the solar wind - Initial results from HELIOS 1, *J. Geophys. Res.*, , *82*, 632–650, doi:10.1029/JA082i004p00632.
- Hess, S. L. G., D. M. Malaspina, and R. E. Ergun (2010), Growth of the Langmuir cavity eigenmodes in the solar wind, *Journal of Geophysical Research (Space Physics)*, *115*(A14), A10103, doi:10.1029/2009JA015179.
- Holman, G. D., M. J. Aschwanden, H. Aurass, M. Battaglia, P. C. Grigis, E. P. Kontar, W. Liu, P. Saint-Hilaire, and V. V. Zharkova (2011), Implications of X-ray Observations for Electron Acceleration and Propagation in Solar Flares, *Space Sci. Rev.* , *159*, 107–166, doi:10.1007/s11214-010-9680-9.
- Kontar, E. P., J. C. Brown, A. G. Emslie, W. Hajdas, G. D. Holman, G. J. Hurford, J. Kašparová, P. C. V. Mallik, A. M. Massone, M. L. McConnell, M. Piana, M. Prato, E. J. Schmahl, and E. Suarez-Garcia (2011a), Deducing Electron Properties from Hard X-ray Observations, *Space Sci. Rev.* , *159*, 301–355, doi:10.1007/s11214-011-9804-x.
- Kontar, E. P., I. G. Hannah, and N. H. Bian (2011b), Acceleration, Magnetic Fluctuations, and Cross-field Transport of Energetic Electrons in a Solar Flare Loop, *Astrophys. J.*, , *730*, L22+, doi:10.1088/2041-8205/730/2/L22.
- Kontar, E. P., H. Ratcliffe, and N. H. Bian (2012), Wave-particle interactions in non-uniform plasma and the interpretation of hard X-ray spectra in solar flares, *Astron. Astrophys.*, , *539*, A43, doi:10.1051/0004-6361/201118216.

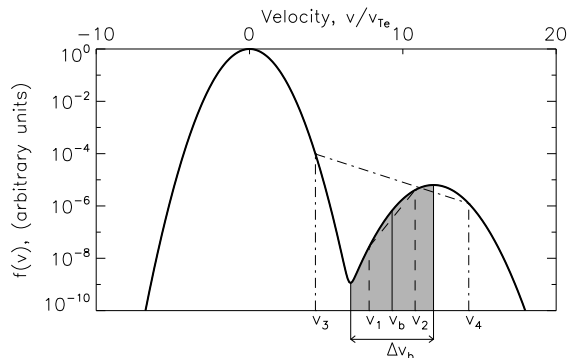
- Krommes, J. A. (2002), Fundamental statistical descriptions of plasma turbulence in magnetic fields, *Phys. Rep.*, *360*, 1–352, doi:10.1016/S0370-1573(01)00066-7.
- Kubo, R. (1954), Note on the Stochastic Theory of Resonance Absorption, *Journal of the Physical Society of Japan*, *9*, 935.
- Kubo, R. (1962), Generalized Cumulant Expansion Method, *Journal of the Physical Society of Japan*, *17*, 1100.
- Lin, R. P. (2011), Energy Release and Particle Acceleration in Flares: Summary and Future Prospects, *Space Sci. Rev.*, *159*, 421–445, doi:10.1007/s11214-011-9801-0.
- Lin, R. P., D. W. Potter, D. A. Gurnett, and F. L. Scarf (1981), Energetic electrons and plasma waves associated with a solar type III radio burst, *Astrophys. J.*, *251*, 364–373, doi:10.1086/159471.
- Lin, R. P., W. K. Levedahl, W. Lotko, D. A. Gurnett, and F. L. Scarf (1986), Evidence for nonlinear wave-wave interactions in solar type III radio bursts, *Astrophys. J.*, *308*, 954–965, doi:10.1086/164563.
- Melrose, D. B. (1980), *Plasma astrophysics. Nonthermal processes in diffuse magnetized plasmas - Vol.1: The emission, absorption and transfer of waves in plasmas; Vol.2: Astrophysical applications*, New York: Gordon and Breach.
- Melrose, D. B. (1985), Plasma emission mechanisms, in *Solar Radiophysics: Studies of Emission from the Sun at Metre Wavelengths*, edited by D. J. McLean and N. R. Labrum, pp. 177–210, Cambridge University Press.
- Melrose, D. B. (2009), Coherent emission, in *IAU Symposium, IAU Symposium*, vol. 257, edited by N. Gopalswamy and D. F. Webb, pp. 305–315, doi:10.1017/S1743921309029470.

- Miller, J. A., P. J. Cargill, A. G. Emslie, G. D. Holman, B. R. Dennis, T. N. LaRosa, R. M. Winglee, S. G. Benka, and S. Tsuneta (1997), Critical issues for understanding particle acceleration in impulsive solar flares, *J. Geophys. Res.*, , *102*, 14,631–14,660, doi:10.1029/97JA00976.
- Mozer, F. S., and C. H. K. Chen (2013), Parallel Electric Field Spectrum of Solar Wind Turbulence, *Astrophys. J.*, , *768*, L10, doi:10.1088/2041-8205/768/1/L10.
- Muschietti, L., M. V. Goldman, and D. Newman (1985), Quenching of the beam-plasma instability by large-scale density fluctuations in 3 dimensions, *Solar Phys.* , *96*, 181–198, doi:10.1007/BF00239800.
- Nishikawa, K., and D. D. Ryutov (1976), Relaxation of relativistic electron beam in a plasma with random density inhomogeneities, *Journal of the Physical Society of Japan*, *41*, 1757–1765.
- Papadopoulos, K., M. L. Goldstein, and R. A. Smith (1974), Stabilization of Electron Streams in Type III Solar Radio Bursts, *Astrophys. J.*, , *190*, 175–186, doi:10.1086/152862.
- Petrosian, V. (2012), Stochastic Acceleration by Turbulence, *Space Sci. Rev.* , *173*, 535–556, doi:10.1007/s11214-012-9900-6.
- Pines, D., and J. R. Schrieffer (1962), Approach to equilibrium of electrons, plasmons, and phonons in quantum and classical plasmas, *Phys. Rev.*, *125*, 804–812, doi:10.1103/PhysRev.125.804.
- Ratcliffe, H., N. H. Bian, and E. P. Kontar (2012), Density Fluctuations and the Acceleration of Electrons by Beam-generated Langmuir Waves in the Solar Corona, *Astrophys. J.*, , *761*, 176, doi:10.1088/0004-637X/761/2/176.

- Robinson, P. A. (1997), Nonlinear wave collapse and strong turbulence, *Reviews of Modern Physics*, *69*, 507–573, doi:10.1103/RevModPhys.69.507.
- Robinson, P. A., and I. H. Cairns (1998), Fundamental and Harmonic Emission in Type III Solar Radio Bursts - I. Emission at a Single Location or Frequency, *Sol. Phys.*, *181*, 363–394.
- Rowland, H. L., and K. Papadopoulos (1977), Simulations of nonlinearly stabilized beam-plasma interactions, *Physical Review Letters*, *39*, 1276–1280, doi:10.1103/PhysRevLett.39.1276.
- Rudakov, L. I., and V. N. Tsytovich (1971), The theory of plasma turbulence for strong wave-particle interaction, *Plasma Physics*, *13*, 213–228, doi:10.1088/0032-1028/13/3/004.
- Sagdeev, R. Z., and A. A. Galeev (1969), *Nonlinear Plasma Theory*, New York: Benjamin.
- Shalchi, A. (Ed.) (2009), *Nonlinear Cosmic Ray Diffusion Theories*, *Astrophysics and Space Science Library*, vol. 362, doi:10.1007/978-3-642-00309-7.
- Stix, T. H. (1992), *Waves in plasmas*, 584 p. pp., New York: American Institute of Physics.
- Sturrock, P. A. (1966), Stochastic Acceleration, *Physical Review*, *141*, 186–191, doi:10.1103/PhysRev.141.186.
- Thornhill, S. G., and D. Ter Haar (1978), Langmuir turbulence and modulational instability, *Phys. Rep.*, *43*, 43–99, doi:10.1016/0370-1573(78)90142-4.
- Tsytovich, V. N. (1995), *Lectures on Non-linear Plasma Kinetics*, 376 pp. pp., Springer-Verlag, Berlin Heidelberg New York.



- Van Vleck, J. H., and H. Margenau (1949), Collision theories of pressure broadening of spectral lines, *Phys. Rev.*, *76*, 1211–1214, doi:10.1103/PhysRev.76.1211.
- Vedenov, A. A., and E. P. Velikhov (1963), Quasilinear Approximation in the Kinetic Theory of a Low-density Plasma, *Soviet Journal of Experimental and Theoretical Physics*, *16*, 682–+.
- Vedenov, A. A., A. V. Gordeev, and L. I. Rudakov (1967), Oscillations and instability of a weakly turbulent plasma, *Plasma Physics*, *9*, 719–735, doi:10.1088/0032-1028/9/6/305.
- Völk, H. J. (1973), Nonlinear Perturbation Theory for Cosmic Ray Propagation in Random Magnetic Fields, *Astrophys. Spa. Sci.*, *25*, 471–490, doi:10.1007/BF00649186.
- Zakharov, V. E. (1972), Collapse of Langmuir Waves, *Soviet Journal of Experimental and Theoretical Physics*, *35*, 908.
- Zaslavsky, A., A. S. Volokitin, V. V. Krasnoselskikh, M. Maksimovic, and S. D. Bale (2010), Spatial localization of Langmuir waves generated from an electron beam propagating in an inhomogeneous plasma: Applications to the solar wind, *Journal of Geophysical Research (Space Physics)*, *115*, A08103, doi:10.1029/2009JA014996.



**Figure 1.** An example of weakening and possible suppression of the beam instability by resonance broadening. The electron distribution  $f(v)$  is the sum of a thermal Maxwellian of temperature  $T_e$  and a beam with velocity  $v_b = 9v_{Te}$  for  $v_{Te}$  the electron thermal velocity. The shaded grey, corresponds to the region of positive slope  $\partial f/\partial v > 0$  in the distribution function. It is centered around  $v_b$  with a width  $\Delta v_b$ . For a resonance width such that  $\Delta v = v_2 - v_1 < \Delta v_b$ , the growth rate  $\gamma(\Delta v) \propto \Delta f/\Delta v > 0$  and, hence, the instability proceeds, but at a slower rate as the effective slope is smaller. For an even larger resonance width, covering  $\Delta v = v_4 - v_3 > \Delta v_b$ , we have  $\gamma(\Delta v) \propto \Delta f/\Delta v < 0$  and, hence, the instability is suppressed, i.e. the beam is stable to the generation of Langmuir waves.